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Generating functions and duality for integer programs

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Abstract

We consider the integer program $\mathbb{P} \to \max\{c'x \mid Ax = y;\, x \in \mathbb{N}^n\}$. Using the generating function of an associated counting problem, and a generalized residue formula of Brion and Vergne, we explicitly relate $\mathbb{P}$ with its continuous linear programming (LP) analogue and provide a characterization of its optimal value. In particular, dual variables $\lambda \in \mathbb{R}^m$ have discrete analogues $z \in \mathbb{C}^m$, related in a simple manner. Moreover, both optimal values of $\mathbb{P}$ and the LP obey the same formula, using z for $\mathbb{P}$ and $|z|$ for the LP. One retrieves (and refines) the so-called group-relaxations of Gomory which, in this dual approach, arise naturally from a detailed analysis of a generalized residue formula of Brion and Vergne. Finally, we also provide an explicit formulation of a dual problem $\mathbb{P}^*$, the analogue of the dual LP in linear programming.

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1. Introduction

With $A \in \mathbb{Z}^{m\times n}$, $c \in \mathbb{R}^n$, $y \in \mathbb{Z}^m$, we consider the integer program

$$\mathbb{P} \to f_c(y) := \max\{c'x \mid Ax = y;\, x \in \mathbb{N}^n\}. \tag{1.1}$$

This discrete analogue of linear programming (LP) is a fundamental NP-hard problem with numerous important applications. Whereas linear programs are solvable in polynomial time, solving $\mathbb{P}$ remains in general a formidable computational challenge. For a standard reference on integer programming, the reader is referred to e.g. [14,16]. The duality results available for integer programs are obtained via the use of *subadditive* functions as in e.g. [21], and the smaller class of *Chvátal* and *Gomory* functions as in e.g. [4] (see also [13] and [16, pp. 346–353] and the many references therein). However, as subadditive, Chvátal and Gomory functions are only defined implicitly from their properties, the resulting dual problems defined in [4] or [21], are essentially conceptual in nature and Gomory functions are rather used to generate valid inequalities for the primal problem.

In some recent new approaches, algebraic methods have been used to characterize and eventually compute the optimal value $f_c(y)$. For instance, the algebraic Conti–Traverso algorithm [7] finds an optimal solution of $\mathbb{P}$ by first computing the reduced Gröbner basis G_c of a toric ideal related to $\mathbb{P}$, with respect to the cost vector c. Then with v any feasible solution of $\mathbb{P}$, one obtains an optimal solution v^* of $\mathbb{P}$ by computing the normal form x^{v^*} of x^v, with respect to G_c. Moreover, through several algebraic

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notions (such as *toric ideal*, *Gröbner basis*, *state polytope*, *Gröbner fan*, etc.), a nice parallel has been established betweeen the discrete problem $\mathbb{P}$ and its continuous analogue, that is, the linear program (LP)

$$g_c(y) := \max\{c'x \mid Ax = y,\ x \geqslant 0;\ x \in \mathbb{R}^n\}. \tag{1.2}$$

In this algebraic approach, integer programming appears as an *arithmetic refinement* of linear programming. For more details the interested reader is referred to [1,9,10,17–19], and the many references therein. In particular, it has been shown how the so-called *group-relaxations* (another older algebraic approach) introduced in the late sixties by Gomory [8] (and later extended in [20]), are related to a fixed *initial ideal*.

On the other hand, our approach developed in [12], which uses the associated *counting* function

$$\widehat{f}_c(y) := \left\{\sum \mathrm{e}^{c'x} \mid Ax = y, x \in \mathbb{N}^n\right\} \tag{1.3}$$

and its *generating function*, also establishes a parallel between $\mathbb{P}$ and the LP dual of (1.2). In a sense, this latter approach is *dual* to the algebraic methods described in [17,19] as well as in [8], for it works in the image space ($\subset \mathbb{Z}^m$) of the linear mapping A instead of the primal space $\mathbb{Z}^n$. The variables of interest associated with the constraints $Ax = y$ are the analogues of the usual *dual* variables in linear programming, except they are now in $\mathbb{C}^m$ rather than in $\mathbb{R}^m$.

Contribution: The goal of this paper is to make this duality statement more precise and provide additional results. Namely:

(a) We extend the results in [12] and relate them to the *group-relaxations* of integer programs. Indeed, we show that these group-relaxations naturally arise from a detailed analysis of a generalized residue formula of Brion and Vergne [5] for the generating function of $\widehat{f}_c(.)$. Our group-relaxations are defined for arbitrary (primal) feasible bases σ (and not only for the optimal basis σ^* of the LP (1.2), as in [8,20]), whereas those defined in [9,10] are defined for (dual) feasible bases σ.

Actually, we need them to explain the case when the group-relaxation associated with σ^* does not provide the optimal value of $\mathbb{P}$, but only an upper bound. In this case, we show that, *necessarily*, there is another primal basis $\sigma \neq \sigma^*$ whose group-relaxation yields the same upper bound. This degenerate case is what we call the discrete analogue of the *nondegeneracy* property in linear programming.

(b) We illustrate our approach on the *knapsack problem*, that is, when $m = 1$, $A \in \mathbb{N}^{1\times n}$, $y \in \mathbb{N}$, and show how the optimal value of $\mathbb{P}$ can be obtained by symbolic calculation.

(c) We make a detailed comparison between the integer program $\mathbb{P}$ and the LP (1.2) from a *dual point of view*. We show that to the optimal solution λ^* of the dual of the LP (1.2), correspond s vectors $z_g = \mathrm{e}^{\lambda^*}\mathrm{e}^{2\mathrm{i}\pi\theta_g}$ in $\mathbb{C}^m$ (or, $\ln z_g = \lambda^* + 2\mathrm{i}\pi\theta_g$), where s is the determinant of the optimal basis of the LP (1.2), and g belongs to a finite group. In other words, z_g is the *periodic analogue* of λ^*. Moreover, for each basis σ of the LP (1.2), we introduce a function $r \mapsto R_\sigma(z_g, r)$, $r \in \mathbb{N}$, that we call the *vertex residue function*. Then under the nondegeneracy property mentioned earlier, we have a complete parallel between $\mathbb{P}$ and the LP (1.2). A simple formula that uses the vertex residue function at the optimal basis σ^* of the LP (1.2), gives the optimal value of $\mathbb{P}$, and the same formula also gives the optimal value of the LP (1.2) when $z_g \in \mathbb{C}^m$ is replaced with the vector $|z_g| \in \mathbb{R}^m$ of its component moduli.

So, if in the primal algebraic approaches described in [17,19], integer programming appears as an *arithmetic refinement* of linear programming, in the present dual approach, integer programming appears as a complexification (in $\mathbb{C}^m$) of the associated LP dual (in $\mathbb{R}^m$). That is, restricting the primal LP (in $\mathbb{R}^n$) to the integers $\mathbb{N}^n$, induces relaxing the dual LP (in $\mathbb{R}^m$) to $\mathbb{C}^m$.

(d) This latter statement is clarified and a dual problem $\mathbb{P}^*$ is explicitly defined. It is the analogue of the LP dual of the LP (1.2) in the sense that it is obtained in a similar fashion, by using the analogue of the Fenchel-transform, in which the dual variables z are now in $\mathbb{C}^m$. Again, if z is replaced with its vector $|z|$ of component moduli, we retrieve the usual Fenchel-transform, and thus, the usual LP dual.

The paper is organized as follows. In Section 2, we first introduce the notation, some definitions and some preliminary results. We then present our main result in Section 3 and illustrate the approach on the knapsack problem. Next, Section 4 is devoted to a comparison between linear programming and integer programming from a dual point of view. Finally, the dual problem $\mathbb{P}^*$ is explicitly defined in Section 5. For the sake of clarity of exposition, most proofs are postponed to Section 6, and some auxiliary results are in Appendix A.

2. Notation, definitions and preliminaries

2.1. Notation, definitions

For any vector x and matrix A, the notation x' and A' stands for their respective transpose. For any $z \in \mathbb{C}^m$, $y \in \mathbb{Z}^m$, the notation z^y stands for the monomial

$$z_1^{y_1} \cdots z_m^{y_m}.$$

Let $A \in \mathbb{Z}^{m\times n}$ be the matrix

$$A = [A_1|\ldots|A_n],$$

where $A_j \in \mathbb{R}^m$ denotes the jth column of A for all $j = 1, \ldots, n$. Thus, for $z \in \mathbb{C}^m$,

$$z^{A_j} = z_1^{A_{1j}} \cdots z_m^{A_{mj}}, \quad j = 1, \ldots, n.$$

For $y \in \mathbb{R}^m$, we denote by $\Omega(y) \subset \mathbb{R}^n$, the convex polyhedron

$$\Omega(y) = \{x \in \mathbb{R}^n \mid Ax = y; x \geqslant 0\}. \tag{2.1}$$

We next introduce some notation taken from Brion and Vergne [5]. With $\Delta := (A_1, \ldots, A_n)$, let $C(\Delta) \subset \mathbb{R}^m$ be the closed convex cone generated by Δ.

Define Λ to be the *lattice* $A(\mathbb{Z}^n)$. A subset σ of $\{1, \ldots, n\}$ is called a *basis* of Δ if the sequence $\{A_j\}_{j\in\sigma}$ is a basis of $\mathbb{R}^m$, and the set of bases of Δ is denoted by $\mathscr{B}(\Delta)$. For $\sigma \in \mathscr{B}(\Delta)$, let $C(\sigma)$ be the cone generated by $\{A_j\}_{j\in\sigma}$. With any $y \in C(\Delta)$ associate the intersection of all cones $C(\sigma)$ which contain y. It defines a subdivision of $C(\Delta)$ into polyhedral cones. The interiors of the maximal cones in this subdivision are called *chambers* in [2]. For every y in a chamber γ, the convex polyhedron $\Omega(y)$ is *simple*. Next, for a chamber γ (whose closure is denoted by $\overline{\gamma}$), let $\mathscr{B}(\Delta, \gamma)$ be the set of bases σ such that γ is contained in $C(\sigma)$, and let $\mu(\sigma)$ denote the volume of the convex polytope $\{\sum_{j\in\sigma} t_j A_j \mid 0 \leqslant t_j \leqslant 1\}$, normalized so that $\mathrm{vol}(\mathbb{R}^m/\Lambda) = 1$.

Observe that for $y \in \overline{\gamma}$ and $\sigma \in \mathscr{B}(\Delta, \gamma)$ we have $y = \sum_{j\in\sigma} x_j(\sigma)A_j$ for some $x_j(\sigma) \geqslant 0$. Therefore, the vector $x(\sigma) \in \mathbb{R}^n_+$ with $x_j(\sigma) = 0$ whenever $j \notin \sigma$, is a *vertex* of the polytope $\Omega(y)$. Denote by V the subspace $\{x \in \mathbb{R}^n \mid Ax = 0\}$. Finally, given $\sigma \in \mathscr{B}(\Delta)$, let $\pi^\sigma \in \mathbb{R}^m$ be the row vector that solves $\pi^\sigma A_j = c_j$ for all $j \in \sigma$. A vector $c \in \mathbb{R}^n$ is said to be *regular* if $c_j - \pi^\sigma A_j \neq 0$ for all $\sigma \in \mathscr{B}(\Delta)$ and all $j \notin \sigma$. Observe that in the LP terminology, $c_k - \pi^\sigma A_k$ is the *reduced cost* of the variable x_k with respect to the basis σ.

Let $c \in \mathbb{R}^n$ be regular with $-c$ in the interior of the dual cone $(\mathbb{R}^n_+ \cap V)^*$ so that the LP (1.2) has a finite optimal value.

2.2. Preliminaries

With $\widehat{f_c}(y)$ in (1.3) is associated its generating function $F_c : \mathbb{C}^m \to \mathbb{C}$

$$z \mapsto F_c(z) := \sum_{y\in\mathbb{Z}^m} \widehat{f_c}(y) z^{-y}, \tag{2.2}$$

well defined on the domain

$$|z^{A_j}| > \mathrm{e}^{c_j}, \quad j = 1, \ldots, n, \tag{2.3}$$

which is nonempty whenever $-c \in \int(\mathbb{R}^n_+ \cap V)^*$. In fact, on its domain (2.3), $F_c(z)$ reads

$$F_c(z) = \frac{1}{\prod_{j=1}^n (1 - \mathrm{e}^{c_j} z^{-A_j})}$$

(see e.g. [5]), and

$$\widehat{f_c}(y) = \frac{1}{(2\mathrm{i}\pi)^m} \int_{|z|=\rho} F_c(z)\, z^{y-e_m}\, \mathrm{d}z, \tag{2.4}$$

where $\rho \in \mathbb{R}^m$ satisfies (2.3), and e_m is the all-ones vector $(1, \ldots, 1)$ of $\mathbb{R}^m$.

Generating functions are specially useful to count lattice points in convex polytopes. For recent results in this vein, the interested reader is referred to e.g. [3,5,6,15], and the many references therein.

For a lattice M, let M^* denote the dual lattice. Given $\sigma \in \mathscr{B}(\Delta)$, the finite group $G(\sigma) := (\oplus_{j\in\sigma}\mathbb{Z}A_j)^*/\Lambda^*$ has finitely many characters $\mathrm{e}^{2\mathrm{i}\pi y}$, $y \in \Lambda$. In particular, for all $A_k \notin \sigma$,

$$\mathrm{e}^{2\mathrm{i}\pi A_k}(g) = \mathrm{e}^{2\mathrm{i}\pi\langle A_\sigma^{-1} A_k, g\rangle}.$$

Let $\Omega(y)$ be the convex polyhedron defined in (2.1) with $y \in \Lambda$. Brion and Vergne [5] provide a nice generalized residue formula for $F_c(z)$ and prove that for all $y \in \overline{\gamma} \cap \Lambda$,

$$\begin{aligned}\widehat{f_c}(y) &= \sum_{x(\sigma):\,\text{vertex of }\Omega(y)} \frac{e^{c'x(\sigma)}}{\mu(\sigma)} \sum_{g\in G(\sigma)} \frac{e^{2i\pi y}(g)}{\prod_{k\notin\sigma}(1-e^{-2i\pi A_k}(g)e^{(c_k-\pi^\sigma A_k)})} \\ &= \sum_{x(\sigma):\,\text{vertex of }\Omega(y)} \frac{e^{c'x(\sigma)}}{\mu(\sigma)}\, U_\sigma(y,c)\end{aligned} \tag{2.5}$$

(see [5, (3.4.1), p. 821]). Based on this result, we have shown in [11,12] that if $Ax = y$ has a solution $x \in \mathbb{N}^n$ and if

$$\max_{x(\sigma):\,\text{vertex of }\Omega(y)} \left[c'x(\sigma) + \lim_{r\to\infty} \frac{1}{r}\ln U_\sigma(y, rc)\right] \tag{2.6}$$

is attained at a unique vertex $x(\hat{\sigma})$, then

$$f_c(y) = c'x(\hat{\sigma}) + \lim_{r\to\infty} \frac{1}{r}\ln U_{\hat{\sigma}}(y, rc). \tag{2.7}$$

In addition, the term $\lim_{r\to\infty} 1/r \ln U_{\hat{\sigma}}(rc, y)$ is shown to be a sum of certain *reduced costs* $c_k - \pi^\sigma A_k$, $k \notin \hat{\sigma}$. We also have the following asymptotic result. For $t \in \mathbb{N}$ sufficiently large,

$$f_c(ty) - g_c(ty) = \lim_{r\to\infty} \frac{1}{r}\ln U_{\sigma^*}(y, rc), \tag{2.8}$$

where σ^* is the optimal basis of the LP (1.2), and so, $f_c(ty) - g_c(ty)$ is a *periodic* (constant) function of period $\mu(\sigma^*) = \det(A_{\sigma^*})$. To prove the above results, one essentially uses

$$e^{f_c(y)} = \lim_{r\to\infty} \widehat{f_{rc}}(y)^{1/r}. \tag{2.9}$$

So, in principle, one may compute the optimal value of $\mathbb{P}$ by evaluating $\widehat{f_{rc}}(y)$ for r sufficiently large. However, this evaluation which requires manipulating complex numbers and exponentials, is nontrivial and numerically ill-posed. However, as we shall see, it provides new insights into $\mathbb{P}$.

3. Main result

We first present our main result in Section 3.1 and then illustrate the whole approach on the knapsack problem in Section 3.3.

3.1. The optimal value of $\mathbb{P}$

We first refine characterization (2.5). We assume for convenience that $c \in \mathbb{Q}^n$, but the result still holds for $c \in \mathbb{R}^n$ (see Remark 3.2).

Let $y \in \Lambda$. Given a vertex $x(\sigma)$ of $\Omega(y)$, define

$$S_\sigma := \{k \notin \sigma \mid A_k \notin \oplus_{j\in\sigma} A_j\mathbb{Z}\} \tag{3.1}$$

and

$$\begin{aligned} M_\sigma^+ &:= \{k \notin \sigma \mid (c_k - \pi^\sigma A_k) > 0\}, \\ M_\sigma^- &:= \{k \notin \sigma \mid (c_k - \pi^\sigma A_k) < 0\}. \end{aligned} \tag{3.2}$$

When c is regular, then M_σ^+, M_σ^- define a partition of $\{1,\dots,n\}\setminus\sigma$. Note also that for the (unique) optimal vertex $x(\sigma^*)$ of LP (1.2) we have $M_{\sigma^*}^+ = \emptyset$. Finally, for every $k \in S_\sigma$, denote by $s_{k\sigma} \in \mathbb{N}$ the smallest integer such that $s_{k\sigma}A_k \in \oplus_{j\in\sigma} A_j\mathbb{Z}$.

Lemma 3.1. *Let $c \in \mathbb{Q}^n$ be regular with $-c \in \int(\mathbb{R}_+^n \cap V)^*$, and $y \in \Lambda$. Let $q \in \mathbb{N}$ be large enough to ensure that $qc \in \mathbb{Z}^m$ and $q(c_k - \pi^\sigma A_k) \in \mathbb{Z}$ for all $\sigma \in \mathscr{B}(\Delta)$, $k \notin \sigma$, and let $u := e^{r/q}$, $r \in \mathbb{R}$. Let $U_\sigma(y, c)$ be as in* (2.5).

Then:
(a) $U_\sigma(y, rc)$ *can be written as*,

$$U_\sigma(y, rc) = \frac{P_{\sigma y}}{Q_{\sigma y}} \tag{3.3}$$

for two Laurent polynomials $P_{\sigma y}, Q_{\sigma y} \in \mathbb{R}[u, u^{-1}]$. *In addition, the maximal algebraic degree of* $P_{\sigma y}$ *is the optimal value of the integer program*

$$q \times \begin{cases} \max & \sum_{k \in S_\sigma} (c_k - \pi^\sigma A_k) x_k \\ s.t. & A_\sigma v + \sum_{k \in S_\sigma} A_k x_k = y, \\ & v \in \mathbb{Z}^m;\, x_k \in \mathbb{N};\, x_k < s_{k\sigma} \quad \forall k \in S_\sigma, \end{cases} \tag{3.4}$$

whereas the maximal algebraic degree of $Q_{\sigma y}$ *is given by*:

$$q \sum_{\substack{k \notin S_\sigma \\ k \in M_\sigma^+}} (c_k - \pi^\sigma A_k) + q \sum_{\substack{k \in S_\sigma \\ k \in M_\sigma^+}} s_{k\sigma}(c_k - \pi^\sigma A_k). \tag{3.5}$$

(b) *As a function of the variable* $u := \mathrm{e}^{r/q}$, *and when* $r \to \infty$,

$$\mathrm{e}^{rc' x(\sigma)} U_\sigma(y, rc) \approx u^{qc' x(\sigma) + \deg(P_{\sigma y}) - \deg(Q_{\sigma y})} \tag{3.6}$$

where "deg" *denotes the algebraic degree* (*i.e., the largest power, sign included*).

For a proof see Section 6.1.

Remark 3.2. Lemma 3.1 is also valid if $c \in \mathbb{R}^n$ instead of $c \in \mathbb{Q}^n$. But this time, $P_{\sigma y}$ and $Q_{\sigma y}$ in (3.3) are not Laurent polynomials anymore.

As a consequence of Lemma 3.1, we obtain:

Corollary 3.3. *Let c be regular with* $-c \in \int(\mathbb{R}_+^n \cap V)^*$, $y \in \Lambda$, *and let* $x(\sigma)$ *be a vertex of* $\Omega(y)$. *Then with* $U_\sigma(y, c)$ *as in* (2.5),

$$c' x(\sigma) + \lim_{r \to \infty} \frac{1}{r} \ln U_\sigma(y, rc) = c' x(\sigma) + \max \mathrm{IP}_\sigma(y), \tag{3.7}$$

where $\max \mathrm{IP}_\sigma(y)$ *is the optimal value of the integer program*

$$\mathrm{IP}_\sigma(y) \begin{cases} \max & \sum_{k \in M_\sigma^+} (c_k - \pi^\sigma A_k)(x_k - s_{k\sigma}) + \sum_{k \in M_\sigma^-} (c_k - \pi^\sigma A_k) x_k \\ s.t. & A_\sigma v + \sum_{k \notin \sigma} A_k x_k = y, \\ & v \in \mathbb{Z}^m;\, x_k \in \mathbb{N},\, x_k < s_{k\sigma} \quad \forall k \notin \sigma. \end{cases} \tag{3.8}$$

For a proof see Section 6.2. When σ is the optimal basis σ^* of LP (1.2), the integer program $\mathrm{IP}_\sigma(y)$ in Corollary 3.3, which, in this case, reads

$$\mathrm{IP}_{\sigma^*}(y) \begin{cases} \max & \sum_{k \notin \sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k \\ \text{s.t.} & A_{\sigma^*} v + \sum_{k \notin \sigma^*} A_k x_k = y, \\ & v \in \mathbb{Z}^m;\, x_k \in \mathbb{N},\, x_k < s_{k\sigma^*} \quad \forall k \notin \sigma^* \end{cases}$$

(because $M_{\sigma^*}^+ = \emptyset$), is the so-called *group-relaxation* introduced in [8], also used in [20], and generalized in [9,10]. Observe that all the variables x_k, $k \notin S_{\sigma^*}$, must be zero because of $x_k < s_{k\sigma^*} = 1$.

The constraint $x_k < s_{k\sigma^*}$ in $\mathrm{IP}_{\sigma^*}(y)$ can be removed. Indeed, if in a solution (v, x) of $\mathrm{IP}_{\sigma^*}(y)$ some variable x_k can be written as $p s_{k\sigma^*} + r_k$ for some $p, r_k \in \mathbb{N}$ with $p > 0$, then one may replace x_k with $\tilde{x}_k := r_k$ and obtain a bettter value because $(c_k - \pi^{\sigma^*} A_k) < 0$ and $p A_k s_{k\sigma^*} = A_{\sigma^*} w$ for some $w \in \mathbb{Z}^m$.

Observe that for every basis $\sigma \neq \sigma^*$ of LP (1.2), max $\mathrm{IP}_\sigma(y) < 0$ and thus,

$$c'x(\sigma) + \max \mathrm{IP}_\sigma(y) < c'x(\sigma), \quad \sigma \neq \sigma^*. \tag{3.9}$$

Corollary 3.3 shows how this group-relaxation concept naturally arises from a dual point of view that considers the counting function $\widehat{f_c}(y)$. Here, and as in [9,10], it is not defined only for the optimal basis σ^* of LP (1.2), as in [8,20]. Our group-relaxations $\mathrm{IP}_\sigma(y)$ are defined for all feasible bases σ of LP (1.2), whereas the extended group-relaxations in [10,19] are defined with respect to the feasible bases σ of the *dual* LP of (1.2). So our former *primal group-relaxations* are bounded because of the constraint $x_k < s_{k\sigma}$ for all $k \in M_\sigma^+$, whereas the latter *dual group-relaxations* of Hosten and Thomas are bounded because $(c_k - \pi^\sigma \sigma A_k) < 0$ for all $k \notin \sigma$, i.e., $M_\sigma^+ = \emptyset$; therefore, and because $M_\sigma^+ = \emptyset$, the latter do not include the bound constraints $x_k < s_{k\sigma}$, and the cost function does not include the term $-(c_k - \pi^\sigma A_k)s_{k\sigma}$. In addition, in the *extended* dual group-relaxations of Hosten and Thomas [10] associated with a basis σ, one enforces the nonnegativity of x_k for some indices$k \in \sigma$ (as in the extended group-relaxations of Wolsey [20] for the optimal basis σ^*). Finally, note that the bound constraint $x_k < s_{k\sigma}$ in (3.8) is not added artificially; it comes from a detailed analysis of the leading term of the rational fraction $P_{\sigma y}(u)/Q_{\sigma y}(u)$ in (3.3), as $u \to \infty$. In particular, the constant term $\sum_{k \in M_\sigma^+} (c_k - \pi^\sigma A_k)s_{k\sigma}$ is the degree of the leading term of $Q_{\sigma y}(u)$; see (3.5) ($s_{k\sigma} = 1$ if $k \notin S_\sigma$). This is why, looking at the leading power in (3.6), this term appears with a minus sign in (3.8).

Of course one may also define what we would call *extended* primal group-relaxations, that is, primal group-relaxations $\mathrm{IP}_\sigma(y)$ in (3.8), with *additional* nonnegativity constraints on some components of the vector v. They would be the primal analogues of the extended dual group-relaxations of Hosten and Thomas. However, the analysis of such extended primal group-relaxations is beyond the scope of the present paper. But roughly speaking, enforcing nonnegativity conditions on some components of the vector v in (3.8), amounts to looking at *nonleading* terms of $P_{\sigma y}(u)/Q_{\sigma y}(u)$ (see Remark 3.4 below).

Remark 3.4. Let us go back to definition (2.5) of $U_\sigma(y, c)$, that is, the compact formula

$$U_\sigma(y, c) = \sum_{g \in G(\sigma)} \frac{\mathrm{e}^{2\mathrm{i}\pi y}(g)}{\prod_{k \notin \sigma}(1 - \mathrm{e}^{-2\mathrm{i}\pi A_k}(g)u^{q(c_k - \pi^\sigma A_k)})}, \tag{3.10}$$

with $u := \mathrm{e}^{1/q}$. Written $U_\sigma(y, c) = P_{\sigma y}(u)/Q_{\sigma y}(u)$, the Laurent polynomial $P_{\sigma y} \in \mathbb{R}[u, u^{-1}]$ *encodes* all of the values v of the feasible solutions of the group-relaxation $\mathrm{IP}_\sigma(y)$ (with no constant term) in the powers of its monomials u^v and the number of solutions with value v, in the coefficient of u^v (see Section 6.1.1). So (3.10) is a compact encoding of the group-relaxation $\mathrm{IP}_\sigma(y)$.

We next obtain:

Theorem 3.5. *Let c be regular with $-c \in \int(\mathbb{R}_+^n \cap V)^*$, and $y \in \Lambda$. Assume that $Ax = y$ has a solution $x \in \mathbb{N}^n$. If the "max" in* (2.6) *is attained at a unique vertex $x(\sigma^*)$ of $\Omega(y)$, then σ^* is an optimal basis of LP* (1.2), *and*

$$\begin{aligned} f_c(y) &= c'x(\sigma^*) + \max \mathrm{IP}_{\sigma^*}(y), \\ &= c'x(\sigma^*) + \left[\begin{array}{l} \max \sum_{k \notin \sigma^*} (c_k - \pi^{\sigma^*} A_k)x_k, \\ A_\sigma v + \sum_{k \notin \sigma^*} A_k x_k = y, \\ v \in \mathbb{Z}^m;\ x_k \in \mathbb{N},\ x_k < s_{k\sigma^*} \quad \forall k \notin \sigma^*. \end{array} \right. \end{aligned} \tag{3.11}$$

Equivalently, the gap between the discrete and continuous optimal values is given by

$$f_c(y) - g_c(y) = \max \mathrm{IP}_{\sigma^*}(y). \tag{3.12}$$

Proof. Let σ^* be an optimal basis of LP (1.2), with corresponding optimal solution $x(\sigma^*) \in \mathbb{R}_+^n$. Let x^* be an optimal solution of $\mathbb{P}$ and let $v_{\sigma^*} := (x^*_{\sigma^*_1}, \ldots, x^*_{\sigma^*_m}) \in \mathbb{N}^m$. The vector $(v_{\sigma^*}, \{x_k^*\}_{k \notin \sigma^*}) \in \mathbb{N}^n$ is a feasible solution to $\mathrm{IP}_{\sigma^*}(y)$. Moreover,

$$\sum_{j=1}^n c_j x_j^* = c'x(\sigma^*) + \sum_{j \notin \sigma^*} (c_k - \pi^{\sigma^*} A_k)x_k^*. \tag{3.13}$$

Therefore, if the "max" in (2.6) is attained at a unique vertex, by (2.6)–(2.7),

$$
\begin{aligned}
f_c(y) &= \max_{x(\sigma):\text{ vertex of }\Omega(y)} \left[c'x(\sigma) + \lim_{r\to\infty} \frac{1}{r} \ln U_\sigma(y, rc) \right] \\
&\geqslant c'x(\sigma^*) + \lim_{r\to\infty} \frac{1}{r} \ln U_{\sigma^*}(y, rc) \\
&= c'x(\sigma^*) + \max \mathrm{IP}_{\sigma^*}(y) \\
&\geqslant c'x(\sigma^*) + \sum_{k\notin\sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k^* \\
&= \sum_{j=1}^{n} c_j x_j^* = f_c(y)
\end{aligned}
$$

and so the "max" in (2.6) is necessarily attained at $\sigma = \sigma^*$. □

As noted in [8], when the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$ provides an optimal solution $x^* \in \mathbb{N}^n$ of $\mathbb{P}$, then x^* is obtained from an optimal solution $x(\sigma^*)$ of LP (1.2), and a *periodic* correction term $\sum_{k\notin\sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k^*$. Indeed, for all $\tilde{y} := y + A_{\sigma^*} v$, $v \in \mathbb{Z}^m$, the group-relaxation $\mathrm{IP}_{\sigma^*}(\tilde{y})$ has same optimal value as $\mathrm{IP}_{\sigma^*}(y)$.

We also obtain the following sufficient condition on the data of $\mathbb{P}$ to ensure that the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$ provides an optimal solution of $\mathbb{P}$.

Corollary 3.6. *Let c be regular with $-c \in \int(\mathbb{R}^n_+ \cap V)^*$, and $y \in \Lambda$. Let $x(\sigma^*)$ be the optimal vertex of LP* (1.2) *with optimal basis $\sigma^* \in \mathscr{B}(\Delta)$. If*

$$
c'x(\sigma^*) + \sum_{k\notin\sigma^*} (c_k - \pi^{\sigma^*} A_k)(s_{k\sigma^*} - 1) > c'x(\sigma) - \sum_{\substack{k\notin\sigma \\ k\in M_\sigma^+}} (c_k - \pi^{\sigma} A_k) \tag{3.14}
$$

for every vertex $x(\sigma)$ of $\Omega(y)$, then the "max" in (2.6) *is attained at $\sigma = \sigma^*$ and*

$$
f_c(y) = c'x(\sigma^*) + \max \mathrm{IP}_{\sigma^*}(y). \tag{3.15}
$$

Proof. The result follows because the left-hand side of (3.14) is a lower bound on $\mathrm{IP}_{\sigma^*}(y)$ whereas the right-hand side is an upper bound on the optimal value of $\mathrm{IP}_\sigma(y)$. □

Note that for $t \in \mathbb{N}$ sufficiently large, and $y := ty$, condition (3.14) is certainly true. So for sufficiently large $t \in \mathbb{N}$, the optimal value of $\mathbb{P}$ (where $Ax = ty$) is given by (3.15).

3.2. The nondegeneracy property

When the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$ does not provide an optimal solution of $\mathbb{P}$, Theorem 3.5 states that *necessarily*, the "max" in (2.6) is attained at *several* bases σ. It is not just related to the fact that at some optimal solution (v, x) of $\mathrm{IP}_{\sigma^*}(y)$, the vector $v \in \mathbb{Z}^m$ has some negative components. There is at least *another* group-relaxation with basis $\sigma \neq \sigma^*$, and such that $c'x(\sigma) + \max \mathrm{IP}_\sigma(y)$ is also a maximum in (2.6).

On the other hand, when the uniqueness property of the "max" in (2.6) holds, then one may qualify σ^* as the unique *optimal basis of the integer program*, in the sense that the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$ is the only one to provide this "max" (and hence, an optimal solution of $\mathbb{P}$). Equivalently, the uniqueness of the "max" in (2.6) is also the uniqueness of the "max" in

$$
\max_{x(\sigma):\text{ vertex of }\Omega(y)} [c'x(\sigma) + \max \mathrm{IP}_\sigma(y)].
$$

This uniqueness property of the "max" in (2.6) is the discrete analogue of the linear programming *nondegeneracy* property. Indeed, an optimal basis is not unique if and only if the optimal vertex of the dual is *degenerate* (that is, there are two different optimal bases of the dual LP with same vertex; observe that when $y \in \gamma$, then $\Omega(y)$ is a *simple* polyhedron (see [5, Proposition p. 818]) and the nondegeneracy property holds for the primal).

Thus, when $y \in \gamma$ (so that the optimal vertex $x^* \in \mathbb{R}^n_+$ is nondegenerate), and $c \in \mathbb{R}^n$ is regular, then σ^* is the unique optimal basis of LP (1.2). As $A_k \in \oplus_{j\in\sigma} A_j \mathbb{R}$ for all $k \notin \sigma$, we may set $s_{k\sigma} = 1$ for all $k \notin \sigma$, and all σ, so that the LP

$$\mathrm{LP}_{\sigma^*}(y) \begin{cases} \max & \sum_{j\notin\sigma^*} (c_k - \pi^{\sigma^*} A_k) x_k \\ \text{s.t.} & A_{\sigma^*} v + \sum_{k\notin\sigma^*} A_k x_k = y, \\ & v \in \mathbb{R};\, x_k \in \mathbb{R},\, 0 \leqslant x_k \leqslant 1 \quad \forall\, k \notin \sigma^* \end{cases} \tag{3.16}$$

is the exact continuous analogue of the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$. Its optimal value is 0 because the cost vector has negative coefficients, and its optimal solution is the optimal solution $x^* \in \mathbb{R}^n_+$ of LP (1.2) (take $v = A_{\sigma^*}^{-1} y \geqslant 0$ and $x^*_k = 0$, for all $k \notin \sigma^*$). Thus,

$$c' x(\sigma^*) + \max\, \mathrm{LP}_{\sigma^*}(y) = c' x(\sigma^*) = g_c(y),$$

exactly as (3.11) for $\mathbb{P}$ in Theorem 3.5.

Moreover, for a basis $\sigma \neq \sigma^*$ of LP (1.2), the LP

$$\mathrm{LP}_{\sigma}(y) \begin{cases} \max & \sum_{k\in M_\sigma^+} (c_k - \pi^{\sigma} A_k)(x_k - 1) + \sum_{k\in M_\sigma^-} (c_k - \pi^{\sigma} A_k) x_k \\ \text{s.t.} & A_{\sigma} v + \sum_{k\notin\sigma} A_k x_k = y, \\ & v \in \mathbb{R};\, x_k \in \mathbb{R},\, 0 \leqslant x_k \leqslant 1 \quad \forall\, k \notin \sigma \end{cases} \tag{3.17}$$

has also optimal value 0 (because $c_k - \pi^\sigma A_k < 0$ whenever $k \in M_\sigma^-$), and

$$c' x(\sigma) + \max\, \mathrm{LP}_{\sigma}(y) = c' x(\sigma) < g_c(y).$$

Therefore, uniqueness of the optimal basis σ^* (when c is regular, and the optimal vertex x^* is nondegenerate) is equivalent to the uniqueness of the "max" in

$$\max_{x(\sigma):\ \text{vertex of}\ \Omega(y)} [c' x(\sigma) + \max\, \mathrm{LP}_\sigma(y)], \tag{3.18}$$

which is also attained at the unique basis σ^*.

Therefore, it makes sense to state the following.

Definition 3.7. Let $c \in \mathbb{R}^n$ be regular and $y \in \Lambda$. The integer program $\mathbb{P}$ has a unique optimal basis if the "max" in (2.6), or, equivalently, the "max" in

$$\max_{x(\sigma):\ \text{vertex of}\ \Omega(y)} [c' x(\sigma) + \max\, \mathrm{IP}_\sigma(y)] \tag{3.19}$$

is attained at a unique basis σ^* (in which case σ^* is the optimal basis of LP (1.2)).

Note that, when c is regular and $\Omega(y)$ is a simple polyhedron, LP (1.2) has a unique optimal basis σ^* which may not be an optimal basis of the integer program $\mathbb{P}$, i.e. the max in (3.19) is not attained at a unique σ (see Example 3.8). In other words, the *nondegeneracy property* for integer programming is a stronger condition than the nondegeneracy property in linear programming.

To see what happens in the case of multiple maximizers σ in (2.6), consider the following elementary knapsack example.

Example 3.8. Let $A := [2, 7, 1] \in \mathbb{Z}^{1\times 3}$, $c = (5, 17, 1) \in \mathbb{N}^3$, $y = 5$. The optimal value $f_c(y)$ is 11 with optimal solution $x^* = (2, 0, 1)$. However, with $\sigma^* = \{1\}$, $A_{\sigma^*} = [2]$, the group-relaxation

$$\mathrm{IP}_{\sigma^*}(y) \to \begin{cases} \max (17 - 35/2) x_2 + (1 - 5/2) x_3, \\ 2v + 7x_2 + x_3 = 5, \\ v \in \mathbb{Z};\, x_2, x_3 \in \mathbb{N},\ x_2, x_3 < 2 \end{cases}$$

has optimal value $-\frac{1}{2}$ at $x_1 = -1, x_2 = 1, x_3 = 0$. Therefore, $c' x(\sigma^*) - \frac{1}{2} = 5 \times \frac{5}{2} - \frac{1}{2} = 12$. On the other hand, let $\sigma := \{2\}$ with $A_\sigma = [7]$. The group-relaxation

$$\mathrm{IP}_{\sigma}(y) \to \begin{cases} \max (5 - 34/7)(x_1 - 7) + (1 - 17/7) x_3, \\ 2x_1 + 7v + x_3 = 5, \\ v \in \mathbb{Z};\, x_1, x_3 \in \mathbb{N},\ x_1, x_3 < 7 \end{cases}$$

has optimal value $-\frac{1}{7}$ at $x_1 = 6$, $x_3 = 0$, $v = -1$, and thus $c'x(\sigma) - \frac{1}{7} = 17 \times \frac{5}{7} - \frac{1}{7} = \frac{84}{7} = 12$. In Lemma 3.1(b), as $r \to \infty$, we have

$$e^{rc'x(\sigma^*)} U_{\sigma^*}(y, rc) \approx u^{12q} \quad \text{and} \quad e^{rc'x(\sigma)} U_{\sigma}(y, rc) \approx -u^{12q}$$

and in fact, these two terms have same coefficient but with opposite sign and thus cancel in the evaluation of $\lim_{r\to\infty} f_{rc}(y)^{1/r}$ in (2.9).

Thus, in this case, $\lim_{r\to\infty} f_{rc}(y)^{1/r}$ is not provided by the leading term of $e^{rc'x(\sigma^*)} U_{\sigma^*}(y, rc)$ as a function of $u = e^r$. We need to examine smaller powers of $P_{\sigma y}$ for all σ.

Had we $c_2 = 16$ instead of $c_2 = 17$, then $\mathrm{IP}_{\sigma^*}(y)$ and $\mathrm{IP}_{\sigma}(y)$ would have $-\frac{3}{2}$ and $-\frac{3}{7}$ as respective optimal values, and with same optimal solutions as before. Thus, again,

$$c'x(\sigma^*) - 3/2 = (25 - 3)/2 = 11, \qquad c'x(\sigma) - 3/7 = (80 - 3)/7 = 11,$$

have same value 11, which is now the optimal value $g_c(y)$. But first observe that the optimal solution x^* of $\mathbb{P}$ is also an optimal solution of $\mathrm{IP}_{\sigma^*}(y)$. Moreover, this time

$$\frac{1}{\mu(\sigma^*)} e^{rc'x(\sigma^*)} U_{\sigma^*}(y, rc) \approx 2u^{11q},$$

because the integer program $\mathrm{IP}_{\sigma^*}(y)$ has two optimal solutions. (See (6.9) in Section 6.1.1 and (6.13) in Section 6.1.2 for the respective coefficients of the leading monomials of $P_{\sigma y}(u)$ and $Q_{\sigma y}(u)$.)

On the other hand,

$$\frac{1}{\mu(\sigma)} e^{rc'x(\sigma)} U_{\sigma}(y, rc) \approx -u^{11q}.$$

Therefore, both $c'x(\sigma^*) + \max\ \mathrm{IP}_{\sigma^*}(y)$ *and* $c'x(\sigma) + \max\ \mathrm{IP}_{\sigma}(y)$ provide the optimal value $f_c(y)$ in (2.6). In this example, the uniqueness property in Definition 3.7 does *not* hold because the "max" in (3.19) is not attained at a unique σ. However, note that LP (1.2) has a unique optimal basis σ^*.

We have seen that the optimal value $f_c(y)$ may not be provided by the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$ when the "max" in (2.6) is attained at several bases σ (let Γ be the set of such bases). This is because, as a function of $e^{r/q}$, the leading monomials of $\mu(\sigma)^{-1} e^{c'x(\sigma)} U_{\sigma}(y, rc)$, $\sigma \in \Gamma$, have coefficients with different signs which permits their possible cancellation in the evaluation of $\lim_{r\to\infty} \widehat{f_{rc}}(y)$ in (2.9). The coefficient of the leading monomial of $P_{\sigma y}(y)$ is positive $(=\mu(\sigma))$ whereas the coefficient of the leading monomial of $Q_{\sigma y}$ is given by $(-1)^{a_\sigma}$, where $a_\sigma = |M_\sigma^+|$ (see Section 6.1.2). For instance, if we list the vertices of LP (1.2) in decreasing order according to the value of $e^{c'x}$, then the second vertex may induce a cancellation, because its corresponding leading monomial has a negative coefficient (since M_σ^+ is a singleton).

3.3. The knapsack problem

We here consider the so-called knapsack problem, that is, when $m = 1$, $A \in \mathbb{N}^{1\times n} = \{a_j\}$, $c \in \mathbb{Q}^n$, $y \in \mathbb{N}$. In this case, with $s := \sum_j a_j$, the generating function $F_c(z)$ in (2.2) reads

$$z \mapsto F_c(z) = \frac{1}{\prod_{j=1}^n (1 - e^{c_j} z^{-a_j})} = \frac{z^s}{\prod_{j=1}^n (z^{a_j} - e^{c_j})}, \tag{3.20}$$

which is well-defined provided $|z|^{a_j} > c_j$ for all $j = 1, \ldots, n$. After possible multiplication by an integer, we may and will assume that $c \in \mathbb{N}^n$. If $c \in \mathbb{N}^n$ is regular then, after relabeling if necessary, we have

$$c_1/a_1 > c_2/a_2 > \cdots > c_n/a_n. \tag{3.21}$$

So with $r \in \mathbb{N}$, letting $u := e^r$, the function

$$\frac{F_{rc}(z)}{z} = \frac{z^{s-1}}{\prod_{j=1}^n (z^{a_j} - u^{c_j})}, \tag{3.22}$$

may be decomposed with respect to z, into simpler rational fractions of the form

$$\frac{F_{rc}(z)}{z} = \sum_{j=1}^{n} \frac{P_j(u,z)}{(z^{a_j} - u^{c_j})}, \tag{3.23}$$

where $P_j(u,.) \in \mathbb{R}[z]$ has degree at most $a_j - 1$, and $P_j(.,z)$ is a rational fraction of u. This decomposition can be obtained by symbolic computation.

Next, write

$$P_j(u,z) = \sum_{k=0}^{a_j-1} P_{jk}(u) z^k, \quad j = 1,\dots,n, \tag{3.24}$$

where the $P_{jk}(u)$'s are rational fractions of u, and let $\rho > rc_1/a_1$. We then have

$$\begin{aligned}\widehat{f}_{rc}(y) &= \sum_{j=1}^{n} \sum_{k=0}^{a_j-1} P_{jk}(\mathrm{e}^r) \int_{|z|=\rho} \frac{z^{y+k}}{(z^{a_j} - \mathrm{e}^{rc_j})}\,\mathrm{d}z \\ &= \sum_{j=1}^{n} \sum_{k=0}^{a_j-1} P_{jk}(\mathrm{e}^r) \begin{cases} \mathrm{e}^{rc_j(y+k+1-a_j)/a_j} & \text{if } y+k+1 = 0 \bmod a_j, \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{3.25}$$

Equivalently, letting $y = s_j \bmod a_j$ for all $j = 1,\dots,n$,

$$\widehat{f}_{rc}(y) = \sum_{j=1}^{n} P_{j(a_j-s_j-1)}(\mathrm{e}^r)\mathrm{e}^{rc_j(b-s_j)/a_j}, \tag{3.26}$$

so that

$$f_c(y) = \lim_{r\to\infty} \frac{1}{r} \ln\left[\sum_{j=1}^{n} P_{j(a_j-s_j-1)}(\mathrm{e}^r)\mathrm{e}^{rc_j(b-s_j)/a_j}\right] \tag{3.27}$$

and if the "max" in (2.6) is attained at a unique basis σ^*, then $\sigma^* = \{1\}$, and

$$f_c(y) = c_1(b-s_1)/a_1 + \lim_{r\to\infty} \frac{1}{r} \ln P_{1(a_1-s_1-1)}(\mathrm{e}^r). \tag{3.28}$$

So, if one has computed symbolically the functions $P_{jk}(u)$, it suffices to *read* the power of the leading term of $P_{1(a_1-s_1-1)}(u)$ as $u \to \infty$, to obtain $f_c(y)$ by (3.28).

Example 3.9. Let us go back to Example 3.8 with $A = [2, 7, 1]$, $c = [5, 17, 1]$ and $y = 5$. The optimal basis of the continuous knapsack is $\sigma = \{1\}$ with $A_\sigma = [2]$. Symbolic computation of $P_1(u,z)$ gives

$$P_1(u,z) = \frac{u^6 + u^5}{(u^3-1)(u-1)} + \frac{u^4 + u}{(u^3-1)(u-1)} z$$

and therefore, as $s_1 = 1$,

$$P_{10}(u) = \frac{u^6 + u^5}{(u^3-1)(u-1)}, \tag{3.29}$$

with leading term $u^{6-4} = u^2$, so that with $y = 5$,

$$c_1(y-s_1)/a_1 + \lim_{r\to\infty} \frac{1}{r} \ln P_{1(a_1-s_1-1)}(\mathrm{e}^r) = \frac{5(5-1)}{2} + 6 - 4 = 12.$$

Similarly, with $\sigma = \{2\}$, $A_\sigma = [7]$, $s_2 = 5$, the term $P_{2(7-5-1)}(u)$ is

$$-\frac{u^{14} + u^{15} + u^{17} + u^{18} + u^{20} + u^{21} + u^{23}}{(u^{10}-1)(u-1)}, \tag{3.30}$$

with leading term $-u^{23-11}=-u^{12}$, so that with $y=5$,

$$c_2(y-s_2)/a_2+\lim_{r\to\infty}\frac{1}{r}\,\ln\,P_{1(a_1-s_1-1)}(\mathrm{e}^r)\;=\;0+23-11=12$$

and we explicitly see that the "max" in (2.6) is not unique. Moreover, the two leading terms u^{12} and $-u^{12}$ cancel in (3.27). On the other hand, the next leading term in (3.29) is now $u^{5-4}=u$ whereas the next leading term in (3.30) is now $-u^{21-11}=-u^{10}$, and the optimal value 11 of $\mathbb{P}$ is provided by 10 plus the power of the next leading term in (3.29), i.e. $10+1=11$.

4. A dual comparison between linear and integer programming

Consider LP (1.2). It is well-known that its optimal value $g_c(y)$ (when finite) is also provided by the optimal value of the dual LP

$$\begin{array}{ll}\min & y'\lambda\\ \text{s.t.} & A'\lambda\geqslant c,\\ & \lambda\in\mathbb{R}^m\end{array}\tag{4.1}$$

and if σ^* is an optimal basis of the primal LP (1.2), then an optimal solution $\lambda^*\in\mathbb{R}^m$ of (4.1) is given by the unique solution λ^* of $A'_{\sigma^*}\lambda=c_{\sigma^*}$, where $c_{\sigma^*}\in\mathbb{R}^m$ is the vector $\{c_j\}_{j\in\sigma^*}$.

Given an arbitrary basis $\sigma\in\mathscr{B}(\Delta)$, consider now the system of equations

$$z_1^{A_{1j}}\cdots z_m^{A_{mj}}\;=\;\mathrm{e}^{c_j},\quad j\in\sigma.\tag{4.2}$$

The above system (4.2) has $\rho(\sigma):=\det(A_\sigma)$ solutions $\{z(k)\}_{k=1}^{\rho(\sigma)}$, written

$$z(k)\;=\;\mathrm{e}^{\lambda}\mathrm{e}^{2\mathrm{i}\pi\theta(k)},\quad k=1,\ldots,\rho(\sigma)\tag{4.3}$$

for a vector $\lambda\in\mathbb{R}^m$ and $\rho(\sigma)$ vectors $\{\theta(k)\}$ in $\mathbb{R}^m$.

Indeed, writing $z=\mathrm{e}^{\lambda}\mathrm{e}^{2\mathrm{i}\pi\theta}$ (i.e., the vector $\{e^{\lambda_j}\mathrm{e}^{2\mathrm{i}\pi\theta_j}\}_{j=1}^m$ in $\mathbb{C}^m$ with $|\theta_j|\leqslant 1$), and passing to the logarithm in (4.2), yields

$$A'_\sigma\lambda+\;2\mathrm{i}\pi A'_\sigma\theta\;=\;c_\sigma\tag{4.4}$$

where $c_\sigma\in\mathbb{R}^m$ is the vector $\{c_j\}_{j\in\sigma}$. Thus, $\lambda\in\mathbb{R}^m$ is the unique solution of $A'_\sigma\lambda=c_\sigma$ and θ satisfies

$$A'_\sigma\theta\;\in\mathbb{Z}^m.\tag{4.5}$$

Equivalently, θ belongs to $(\oplus_{j\in\sigma}A_j\mathbb{Z})^*$, the dual lattice of $\oplus_{j\in\sigma}A_j\mathbb{Z}$. Thus, there is a one-to-one correspondence between $\{\theta(k)\}$ and the finite group $G'(\sigma)=(\oplus_{j\in\sigma}A_j\mathbb{Z})^*/\mathbb{Z}^m$. With $G(\sigma)=\{g_1,\ldots,g_s\}$ and $s:=\mu(\sigma)$, define the mapping $\theta:G(\sigma)\to\mathbb{R}^m$

$$g\;\mapsto\;\theta_g:=(A'_\sigma)^{-1}g,$$

so that, for every character $\mathrm{e}^{2\mathrm{i}\pi y}$ of $G(\sigma)$, $y\in\Lambda$, we have

$$\mathrm{e}^{2\mathrm{i}\pi y}(g)\;=\;\mathrm{e}^{2\mathrm{i}\pi y'\theta_g},\quad y\in\Lambda,\;\; g\in G(\sigma)\tag{4.6}$$

and

$$\mathrm{e}^{2\mathrm{i}\pi A_j}(g)\;=\;\mathrm{e}^{2\mathrm{i}\pi A'_j\theta_g}\;=1,\quad j\in\sigma^*.\tag{4.7}$$

For every $\sigma\in\mathscr{B}(\Delta)$, denote by $\{z_g\}_{g\in G(\sigma)}$ these $\mu(\sigma)$ solutions of (4.3), that is,

$$z_g\;=\;\mathrm{e}^{\lambda}\mathrm{e}^{2\mathrm{i}\pi\theta_g},\quad g\in G(\sigma),\tag{4.8}$$

with $\lambda=(A'_\sigma)^{-1}c_\sigma$, and where $\mathrm{e}^{\lambda}\in\mathbb{R}^m$ is the vector $\{\mathrm{e}^{\lambda_i}\}_{i=1}^m$.

So, in LP (1.2) we have a dual vector $\lambda\in\mathbb{R}^m$ associated with each basis σ. In the integer program $\mathbb{P}$, with each (same) basis σ are now associated $\mu(\sigma)$ "dual" vectors $\lambda+2\mathrm{i}\pi\theta_g$, $g\in G(\sigma)$. Hence, with a basis σ in linear programming, the "dual variables" in integer programming are obtained from (a), the corresponding dual variables $\lambda\in\mathbb{R}^m$ in linear programming, and (b), a periodic correction term $2\mathrm{i}\pi\theta_g\in\mathbb{C}^m$, $g\in G(\sigma)$.

We next introduce what we call the *vertex residue function*.

Definition 4.1. Let $y \in \Lambda$ and let $c \in \mathbb{R}^n$ be regular. Let $\sigma \in \mathscr{B}(\Delta)$ be a basis of LP (1.2), and for every $r \in \mathbb{N}$, let $\{z_{gr}\}_{g \in G(\sigma)}$ be as in (4.8) with rc in lieu of c, that is,

$$z_{gr} = \mathrm{e}^{r\lambda}\mathrm{e}^{2\mathrm{i}\pi\theta_g} \in \mathbb{C}^m, \; g \in G(\sigma) \quad (\text{with } \lambda = (A'_\sigma)^{-1}c_\sigma).$$

The vertex residue function associated with the basis σ of LP (1.2), is the function $R_\sigma(z_g, .) : \mathbb{N} \to \mathbb{R}$ defined by:

$$r \mapsto R_\sigma(z_g, r) := \frac{1}{\mu(\sigma)} \sum_{g \in G(\sigma)} \frac{z_{gr}^y}{\prod_{k \notin \sigma}(1 - z_{gr}^{-A_k}\mathrm{e}^{rc_k})}, \tag{4.9}$$

which is well defined because when c is regular, $|z_{gr}|^{A_k} \neq \mathrm{e}^{rc_k}$ for all $k \notin \sigma$.

The name *vertex residue* is now clear because in integration (2.4), $R_\sigma(z_g, r)$ is to be interpreted as a generalized Cauchy residue, with respect to the $\mu(\sigma)$ "poles" $\{z_{gr}\}$ of the generating function $F_{rc}(z)$.

Proposition 4.2. *Let c be regular with $-c \in \int(\mathbb{R}^n_+ \cap V)^*$, and $y \in \Lambda$. Assume that $Ax = y$ has a solution $x \in \mathbb{N}^n$ and the "max" in* (2.6) *is attained at a unique vertex $x(\sigma^*)$ of $\Omega(y)$. Let $\{z_g\}_{g \in G(\sigma^*)}$ be as in* (4.8) *with $\sigma = \sigma^*$. Then*:

(a) *σ^* is an optimal basis of LP* (1.2).
(b) *The optimal value of $\mathbb{P}$ satisfies*

$$\begin{aligned} f_c(y) &= \lim_{r\to\infty} \frac{1}{r} \ln\left[\frac{1}{\mu(\sigma^*)} \sum_{g \in G(\sigma^*)} \frac{z_{gr}^y}{\prod_{k \notin \sigma^*}(1 - z_{gr}^{-A_k}\mathrm{e}^{rc_k})}\right] \\ &= \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(z_g, r) \end{aligned} \tag{4.10}$$

and the optimal value of LP (1.2) *satisfies*

$$\begin{aligned} g_c(y) &= \lim_{r\to\infty} \frac{1}{r} \ln\left[\frac{1}{\mu(\sigma^*)} \sum_{g \in G(\sigma^*)} \frac{|z_{gr}|^y}{\prod_{k \notin \sigma^*}(1 - |z_{gr}|^{-A_k}\mathrm{e}^{rc_k})}\right] \\ &= \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(|z_g|, r). \end{aligned} \tag{4.11}$$

Proof. (a) Follows from Theorem 3.5 and our assumptions.

(b) Let $U_{\sigma^*}(c, y)$ be as in (2.5). It is immediate to see that $\pi^{\sigma^*} = (\lambda^*)'$ and so

$$\mathrm{e}^{-\pi^{\sigma^*}A_k}\mathrm{e}^{-2\mathrm{i}\pi A_k}(g) = \mathrm{e}^{-A'_k\lambda^*}\mathrm{e}^{-2\mathrm{i}\pi A'_k\theta_g} = z_g^{-A_k}, \quad g \in G(\sigma^*).$$

Next, using $c'x(\sigma^*) = y'\lambda^*$,

$$\mathrm{e}^{c'x(\sigma^*)}\mathrm{e}^{2\mathrm{i}\pi y}(g) = \mathrm{e}^{y'\lambda^*}\mathrm{e}^{2\mathrm{i}\pi y'\theta_g} = z_g^y, \quad g \in G(\sigma^*).$$

Therefore,

$$\begin{aligned} \frac{1}{\mu(\sigma^*)}\mathrm{e}^{c'x(\sigma)}U_{\sigma^*}(c, y) &= \frac{1}{\mu(\sigma^*)} \sum_{g \in G(\sigma^*)} \frac{z_g^y}{\prod_{k \notin \sigma^*}(1 - z_g^{-A_k}\mathrm{e}^{c_k})} \\ &= R_{\sigma^*}(z_g, 1) \end{aligned}$$

and (b) follows from (2.7) because, with rc in lieu of c, z_g becomes $z_{gr} = \mathrm{e}^{r\lambda^*}\mathrm{e}^{2\mathrm{i}\pi\theta_g}$ (only the modulus changes).

Next, as only the modulus of z_g is involved in (4.11), we have $|z_{gr}| = \mathrm{e}^{r\lambda^*}$ for all $g \in G(\sigma^*)$, so that

$$\frac{1}{\mu(\sigma^*)} \sum_{g \in G(\sigma^*)} \frac{|z_{gr}|^y}{\prod_{k \notin \sigma^*}(1 - |z_{gr}|^{-A_k}\mathrm{e}^{rc_k})} = \frac{\mathrm{e}^{ry'\lambda^*}}{\prod_{k \notin \sigma^*}(1 - \mathrm{e}^{r(c_k - A'_k\lambda^*)})}$$

Table 1
Comparing $\mathbb{P}$ and LP (1.2)

LP (1.2)	Integer program $\mathbb{P}$
Unique optimal basis σ^* (λ^* nondegenerate)	Unique optimal basis σ^*
One optimal dual vector $\lambda^* \in \mathbb{R}^m$	$\mu(\sigma^*)$ Dual vectors $z_g \in \mathbb{C}^m$, $g \in G(\sigma^*)$
	$\ln z_g = \lambda^* + 2i\pi\theta_g$
$g_c(y) = \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(\lvert z_g\rvert, r)$	$f_c(y) = \lim_{r\to\infty} \frac{1}{r} \ln R_{\sigma^*}(z_g, r)$

and, as $r \to \infty$,

$$\frac{e^{ry'\lambda^*}}{\prod_{k\notin\sigma^*}(1 - e^{r(c_k - A'_k\lambda^*)})} \approx e^{ry'\lambda^*},$$

because $(c_k - A'_k\lambda^*) < 0$ for all $k \notin \sigma^*$. Therefore,

$$\lim_{r\to\infty} \frac{1}{r} \ln\left[\frac{e^{ry'\lambda^*}}{\prod_{k\notin\sigma^*}(1 - e^{r(c_k - A'_k\lambda^*)})}\right] = y'\lambda^* = g_c(y),$$

the desired result. □

Remark 4.3. Proposition 4.2(b) shows that there is indeed a strong relationship between the integer program $\mathbb{P}$ and its continuous analogue, LP (1.2). Both optimal values obey exactly the same formula (4.10), but for the continuous version, the complex vector $z_g \in \mathbb{C}^m$ is replaced with the vector $|z_g| = e^{\lambda^*} \in \mathbb{R}^m$ of its component moduli, where $\lambda^* \in \mathbb{R}^m$ is the optimal solution of the dual LP of (1.2).

Recall from Section 3.2 that when $c \in \mathbb{R}^n$ is regular and $y \in \gamma$, LP (1.2) has a unique optimal basis σ^* (equivalently, the optimal vertex of the dual LP is nondegenerate). For the integer program $\mathbb{P}$, the corresponding uniqueness property (see Definition 3.7) is stronger. To conclude this section, and with this in mind, we have the following correspondence: As summarized in Table 1, from a dual point view, the integer program $\mathbb{P}$ is seen as an extension in $\mathbb{C}^m$ of the dual of LP (1.2) in $\mathbb{R}^m$, whereas in the algebraic primal approaches as described in [8,17,19], $\mathbb{P}$ appears as an arithmetic refinement of LP (1.2).

The next interesting question is: can we provide an explicit description of what would be a *dual* of $\mathbb{P}$? The purpose of the next section is to present such a dual problem $\mathbb{P}^*$.

5. A dual of $\mathbb{P}$

In this section we provide a formulation of a problem $\mathbb{P}^*$, a dual of $\mathbb{P}$, which is the analogue of the LP dual of (1.2).

Recall that when LP (1.2) has finite optimal value $g_c(y)$, we have the well-known *convex duality* result

$$g_c(y) = \inf_{\lambda\in\mathbb{R}^m} y'\lambda + (-g_c)^*(-\lambda), \tag{5.1}$$

where $(-g_c)^* : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$ given by

$$\lambda \mapsto (-g_c)^*(\lambda) := \sup_{y\in\mathbb{R}^m} y'\lambda + g_c(y),$$

is the Fenchel-transform of the convex function $-g_c$. Equivalently,

$$g_c(y) = \inf_{\lambda\in\mathbb{R}^m} y'\lambda + (-g_c)^*(-\lambda) = \min_{\lambda\in\mathbb{R}^m} \{y'\lambda \mid A'\lambda \geqslant c\}, \tag{5.2}$$

and we retrieve the usual LP dual (4.1) of LP (1.2).

But we can also write (5.1) as

$$g_c(y) = \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n_+} \lambda'(y-Ax)+c'x \tag{5.3}$$

or, equivalently,

$$\mathrm{e}^{g_c(y)} = \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n_+} \mathrm{e}^{\lambda'(y-Ax)}\mathrm{e}^{c'x}. \tag{5.4}$$

Define now the following optimization problem:

$$\begin{aligned}\mathbb{P}^* \to \gamma^*(y) &:= \inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\,\mathrm{e}^{c'x}),\\ &= \inf_{z\in\mathbb{C}^m} f^*_c(z,y),\end{aligned} \tag{5.5}$$

where $u \mapsto \Re(u)$ denotes the real part of $u \in \mathbb{C}$.

Clearly, the function $f^*_c : \mathbb{C}^m \times \mathbb{Z}^m \to \mathbb{R}$,

$$\begin{aligned}(z,y) \mapsto f^*_c(z,y) &= \sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\,\mathrm{e}^{c'x})\\ &= \sup_{x\in\mathbb{N}^n} \Re\left(z^y \prod_{j=1}^n (z^{-A_j}\mathrm{e}^{c_j})^{x_j}\right)\end{aligned} \tag{5.6}$$

is finite if and only if $|z^{A_j}| \geqslant \mathrm{e}^{c_j}$ for all $j=1,\ldots,n$, that is, if and only if $A'\ln|z| \geqslant c$, which is the feasible set of the LP dual (4.1).

We claim that $\mathbb{P}^*$ is a dual problem of $\mathbb{P}$. Under an appropriate re-scaling $c \to \tilde{c} := \alpha c$, of the cost vector c, and a condition on the group $G(\sigma^*)$ associated with the optimal basis of LP (1.2), $\mathbb{P}^*$ has the same optimal value as $\mathbb{P}$.

Theorem 5.1. *Let $y \in \Lambda$ and $c \in \mathbb{R}^n$ be regular. Assume that the integer program $\mathbb{P}$ has a feasible solution, and the uniqueness property holds* (*see Definition* 3.7). *Let σ^* be the optimal basis of LP* (1.2), *and let λ^* be the corresponding optimal solution of the dual problem* (4.1).

Assume that there exists $g^* \in G(\sigma^*)$ such that $\mathrm{e}^{2\mathrm{i}\pi y}(g^*) \neq 1$ whenever $y \notin \oplus_{j\in\sigma^*} A_j\mathbb{Z}$.

Let $\tilde{c} := \alpha c$ with $\alpha > 0$. If α is sufficiently small, i.e., $0<\alpha<\overline{\alpha}$, for some $\overline{\alpha} \in \mathbb{R}^+$, then:

(a) *The optimal value $f_c(y)$ of $\mathbb{P}$ satisfies*

$$\begin{aligned}\mathrm{e}^{\alpha f_c(y)} = \mathrm{e}^{f_{\tilde{c}}(y)} = \gamma^*(y) &= \inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\,\mathrm{e}^{\tilde{c}'x})\\ &= \inf_{z\in\mathbb{C}^m} f^*_{\tilde{c}}(z,y).\end{aligned} \tag{5.7}$$

(b) *With $g^* \in G(\sigma^*)$ and $z_{g^*} \in \mathbb{C}^m$ as in* (4.8) (*with $\alpha\lambda^*$ in lieu of λ^**)

$$\begin{aligned}\mathrm{e}^{\alpha f_c(y)} = \mathrm{e}^{f_{\tilde{c}}(y)} = \gamma^*(y) &= \sup_{x\in\mathbb{N}^n} \Re(z_{g^*}^{y-Ax}\,\mathrm{e}^{\tilde{c}'x})\\ &= f^*_{\tilde{c}}(z_{g^*},y).\end{aligned} \tag{5.8}$$

For a proof see Section 6.3. The assumption on g^* in Theorem 5.1 is satisfied in particular when the group $G(\sigma^*)$ is cyclic.

Notice that $\mathbb{P}^*$ becomes the usual LP dual of LP (1.2) when $z \in \mathbb{C}^m$ is replaced with $|z| \in \mathbb{R}^m$. Indeed, from (5.1), we have

$$\begin{aligned}g_c(y) &= \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n_+} \lambda'(y-Ax)+c'x = \inf_{\lambda\in\mathbb{R}^m} \lambda'y + \sup_{x\in\mathbb{R}^n_+} (c'-A'\lambda)x\\ &= \inf_{\lambda\in\mathbb{R}^m} \lambda'y + \sup_{x\in\mathbb{N}^n} (c'-A'\lambda)x = \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{N}^n} \lambda'(y-Ax)+c'x,\end{aligned}$$

that is, one may replace the "sup" over $\mathbb{R}^m_+$ by the "sup" over $\mathbb{N}^n$. Therefore, (5.4) becomes

$$\mathrm{e}^{g_c(y)} = \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{N}^n} \mathrm{e}^{\lambda'(y-Ax)}\mathrm{e}^{c'x}. \tag{5.9}$$

On the other hand, if in (5.5), we replace $z \in \mathbb{C}^m$ by the vector of its component moduli $|z| = e^{\lambda} \in \mathbb{R}^m$, we obtain

$$\inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(|z|^{y-Ax}\, e^{c'x}) = \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{N}^n} e^{\lambda'(y-Ax)}\, e^{c'x} = e^{g_c(y)}. \tag{5.10}$$

Hence, when the uniqueness property (see Definition 3.7) holds for $\mathbb{P}$, Table 1 in Section 4 can be completed by

$$e^{g_c(y)} = \inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(|z|^{y-Ax}\, e^{c'x}) = \inf_{z\in\mathbb{C}^m} f_c^*(|z|, y),$$
$$e^{f_c(y)} = \inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\, e^{c'x}) = \inf_{z\in\mathbb{C}^m} f_c^*(z, y).$$

Again, as for the vertex residue function, there is a complete analogy between $\mathbb{P}^*$ and the dual LP (4.1) (equivalently (5.9)), by just changing $z \in \mathbb{C}^m$ with $|z| \in \mathbb{R}^m$.

6. Proofs

6.1. *Proof of Lemma 3.1*

From (2.5) we have

$$U_\sigma(y, rc) = \sum_{g\in G(\sigma)} \frac{e^{2i\pi y}(g)}{\prod_{k\notin\sigma}(1 - e^{-2i\pi A_k}(g) e^{r(c_k - \pi^\sigma A_k)})} \tag{6.1}$$

and so letting $u := e^{r/q}$ we have

$$U_\sigma(y, rc) = \sum_{g\in G(\sigma)} \frac{e^{2i\pi y}(g)}{\prod_{k\notin\sigma}(1 - e^{-2i\pi A_k}(g) u^{q(c_k - \pi^\sigma A_k)})}. \tag{6.2}$$

Let S_σ be as in (3.1). As $e^{-2i\pi A_k}(g) = 1$ whenever $A_k \in \oplus_{j\in\sigma} A_j \mathbb{Z}$,

$$U_\sigma(y, rc) = \frac{1}{\prod_{k\notin\sigma\cup S_\sigma}(1 - u^{q(c_k - \pi^\sigma A_k)})} \sum_{g\in G(\sigma)} \frac{e^{2i\pi y}(g)}{\prod_{k\in S_\sigma}(1 - e^{-2i\pi A_k}(g) u^{q(c_k - \pi^\sigma A_k)})}, \tag{6.3}$$

which, after reduction to the same denominator, can be written

$$U_\sigma(y, rc) = \frac{P_{\sigma y}}{Q_{\sigma y}}, \tag{6.4}$$

for two Laurent polynomials $P_{\sigma y}, Q_{\sigma y} \in \mathbb{R}[u, u^{-1}]$.

6.1.1. *The Laurent polynomial $P_{\sigma y}(u)$*

Write the finite group

$$G(\sigma) := (\oplus_{j\in\sigma} A_j \mathbb{Z})^* / \Lambda^*$$

as $G(\sigma) = \{g_1, \ldots, g_s\}$ with $s := \mu(\sigma)$, and for $k \in S_\sigma$, consider the character $e^{-2i\pi A_k}$ of the group $G(\sigma)$. For $k \in S_\sigma$, define in $G(\sigma)$ the equivalence relationship

$$g \sim g' \Leftrightarrow e^{-2i\pi A_k}(g) = e^{-2i\pi A_k}(g'), \quad g, g' \in G(\sigma).$$

According to $\sim$, one may partition $G(\sigma)$ into $s_{k\sigma}$ equivalence classes $\{C_i^k\}$ of the same cardinality $s/s_{k\sigma}$, where $s_{k\sigma} \in \mathbb{N}$ is the smallest integer for which $s_{k\sigma} A_k \in \oplus_{j\in\sigma} A_j \mathbb{Z}$ (see Section A.2 for more details). Let $G_k(\sigma)$ be the set of $s_{k\sigma}$ equivalence classes of $G(\sigma)$. A representative of the equivalence class of $g \in G(\sigma)$ is denoted by $\tilde{g}$, so that $G_k(\sigma) := \{\tilde{g}_1, \ldots, \tilde{g}_{s_{k\sigma}}\}$. As $\mathrm{e}^{-2\mathrm{i}\pi A_k}(g) = \mathrm{e}^{-2\mathrm{i}\pi A_k}(g') = \mathrm{e}^{-2\mathrm{i}\pi A_k}(\tilde{g})$ if $g \sim g'$,

$$P_{\sigma y}(u) = \sum_{g \in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(g) \prod_{k \in S_\sigma} \left[\prod_{\tilde{g}' \neq \tilde{g}} (1 - \mathrm{e}^{-2\mathrm{i}\pi A_k}(\tilde{g}') u^{q(c_k - \pi^\sigma A_k)}) \right]. \tag{6.5}$$

Therefore, it follows that $P_{\sigma y}(u)$ is a sum of monomials of the form u^v, $v \in \mathbb{Z}$, where

$$v = q \sum_{k \in S_\sigma} x_k (c_k - \pi^\sigma A_k) \quad \text{with } x_k \in \mathbb{N},\ x_k < s_{k\sigma}, k \in S_\sigma. \tag{6.6}$$

The corresponding coefficient of this monomial u^v of $P_{\sigma y}(u)$ is given by

$$\sum_{g \in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(g) \prod_{k \notin \sigma} \Gamma(g, k),$$

with

$$\Gamma(g,k) = \left\{ \begin{array}{ll} & (-1)^{x_k} \sum \mathrm{e}^{-2\mathrm{i}\pi A_k}(\tilde{g}_{i_1} + \cdots + \tilde{g}_{i_{x_k}}) \\ \text{s.t.} & 1 \leqslant i_1 < i_2 \cdots < i_{x_k} \leqslant s_{k\sigma} \\ & \tilde{g}_{i_1}, \ldots, \tilde{g}_{i_{x_k}} \neq \tilde{g} \end{array} \right\}$$
$$= \mathrm{e}^{-2\mathrm{i}\pi A_k x_k}(\tilde{g}) = \mathrm{e}^{-2\mathrm{i}\pi A_k x_k}(g)$$

(by Lemma A.2(b)).

Hence, the coefficient of the monomial u^v of $P_{\sigma y}(u)$ with v as in (6.6), is

$$\begin{aligned} \sum_{g \in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(g) \prod_{k \in S_\sigma} \Gamma(g,k) &= \sum_{g \in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(g) \prod_{k \in S_\sigma} \mathrm{e}^{-2\mathrm{i}\pi A_k x_k}(g) \\ &= \sum_{g \in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi (y - \sum_{k \in S_\sigma} A_k x_k)}(g) \\ &= \begin{cases} s & \text{if } (y - \sum_{k \in S_\sigma} A_k x_k) \in \oplus_{j \in \sigma} A_j \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{6.7}$$

(see Lemma A.3). Consequently, the maximum algebraic degree of $P_{\sigma y}$ is given by the leading monomial u^v where

$$v = \begin{cases} q \max & \sum_{k \in S_\sigma} (c_k - \pi^\sigma A_k) x_k \\ \text{s.t.} & A_\sigma x_\sigma + \sum_{k \in S_\sigma} A_k x_k = y, \\ & x_k \in \mathbb{N};\ x_k < s_{k\sigma} \quad \forall k \in S_\sigma, \\ & x_\sigma \in \mathbb{Z}^m \end{cases} \tag{6.8}$$

and the coefficient of the monomial u^v of $P_{\sigma y}$ is

$$\mu(\sigma) \times \text{ the number of optimal solutions of (6.8).} \tag{6.9}$$

6.1.2. The Laurent polynomial $Q_{\sigma y}(u)$

For all $k \in S_\sigma$, let $s_{k\sigma}$, $G_k(\sigma)$ be as defined in Section 6.1.1. As $\mathrm{e}^{-2\mathrm{i}\pi A_k}(g)$ is constant in each equivalence class, we may write the Laurent polynomial $Q_{\sigma y}$ as the product $Q^1_{\sigma y} Q^2_{\sigma y}$ of the two Laurent polynomials

$$Q^1_{\sigma y}(u) := \prod_{k \notin \sigma \cup S_\sigma} (1 - u^{q(c_k - \pi^\sigma A_k)}) \tag{6.10}$$

and

$$Q^2_{\sigma y}(u) := \prod_{k \in S_\sigma} \prod_{\tilde{g} \in G_k(\sigma)} (1 - \mathrm{e}^{-2\mathrm{i}\pi A_k}(\tilde{g}) u^{q(c_k - \pi^\sigma A_k)}). \tag{6.11}$$

With arguments similar to those used for $P_{\sigma y}$ one may see that $Q^2_{\sigma y}$ is a Laurent polynomial with all powers of the form u^v, $v \in \mathbb{Z}$, where

$$v = q \sum_{k \in S_\sigma} (c_k - \pi^\sigma A_k) x_k : x_k \in \mathbb{N};\ \ x_k \leqslant s_{k\sigma}.$$

And for the same reasons as for $P_{\sigma y}$, the only monomials u^v with nonzero coefficient are those for which

$$\sum_{k \in S_\sigma} A_k x_k \in \oplus_{j \in \sigma} A_j \mathbb{Z},$$

which is the case if $x_k = s_{k\sigma}$. So the maximum algebraic degree of $Q^2_{\sigma y}$ is obtained with $x_k = s_{k\sigma}$ for all $k \in S_\sigma$ with $(c_k - \pi^\sigma A_k) > 0$, and $x_k = 0$ otherwise. In addition, the coefficient of this monomial is

$$\begin{aligned}\prod_{k \in S_\sigma \cap M^+_\sigma} \left[(-1)^{s_{k\sigma}} \prod_{\tilde{g} \in G_k(\sigma)} \mathrm{e}^{-2\mathrm{i}\pi A_k}(\tilde{g}) \right] &= \prod_{k \in S_\sigma \cap M^+_\sigma} (-1)^{2 s_{k\sigma} + 1} \\ &= (-1)^{|S_\sigma \cap M^+_\sigma|},\end{aligned}$$

where we have used Lemma A.3.

Similarly, the maximum algebraic degree of $Q^1_{\sigma y}$ is given by the sum of $q(c_k - \pi^\sigma A_k)$ over all $k \notin \sigma \cup S_\sigma$ with $c_k - \pi^\sigma A_k > 0$. Therefore, the maximum algebraic degree of $Q_{\sigma y}$ is given by

$$\deg(Q_{\sigma y}) = q \sum_{\substack{k \notin \sigma \cup S_\sigma \\ k \in M^+_\sigma}} (c_k - \pi^\sigma A_k) + q \sum_{\substack{k \in S_\sigma \\ k \in M^+_\sigma}} s_{k\sigma} (c_k - \pi^\sigma A_k). \tag{6.12}$$

(In particular, it is 0 for the optimal basis σ^* of LP (1.2).) Finally, the coefficient of this leading monomial of $Q_{\sigma y}(y)$ is given by $(-1)^{a_\sigma}$ where

$$\begin{aligned} a_\sigma &= |S_\sigma \cap M^+_\sigma| + \left| \{k \notin \sigma \cup S_\sigma;\ k \in M^+_\sigma\} \right| \\ &= |M^+_\sigma|. \end{aligned} \tag{6.13}$$

This completes the proof of Lemma 3.1(a).

(b) (3.6) just identifies the leading term when $r \to \infty$. $\square$

6.2. *Proof of Corollary 3.3*

In view of (6.8) and (6.12)

$$\frac{1}{q}\,[\deg(P_{\sigma y}) - \deg(Q_{\sigma y})] = -\sum_{k \notin S_\sigma; k \in M^+_\sigma} (c_k - \pi^\sigma A_k) + \left\{ \begin{array}{ll} \max & \displaystyle\sum_{k \in S_\sigma \cap M^+_\sigma} (c_k - \pi^\sigma A_k)(x_k - s_{k\sigma}) + \sum_{k \in S_\sigma \cap M^-_\sigma} (c_k - \pi^\sigma A_k) x_k \\ \text{s.t.} & A_\sigma x_\sigma + \displaystyle\sum_{k \in S_\sigma} A_k x_k = y, \\ & x_\sigma \in \mathbb{Z}^m;\ x_k \in \mathbb{N};\ x_k < s_{k\sigma} \quad \forall k \in S_\sigma. \end{array} \right. \tag{6.14}$$

Equivalently,

$$\frac{1}{q}[\deg(P_{\sigma y}) - \deg(Q_{\sigma y})] = \left\{ \begin{array}{ll} \max & \displaystyle\sum_{k \in M^+_\sigma} (c_k - \pi^\sigma A_k)(x_k - s_{k\sigma}) + \sum_{k \in M^-_\sigma} (c_k - \pi^\sigma A_k) x_k \\ \text{s.t.} & A_\sigma x_\sigma + \displaystyle\sum_{k \in S_\sigma} A_k x_k = y, \\ & x_\sigma \in \mathbb{Z}^m;\ x_k \in \mathbb{N};\ x_k < s_{k\sigma} \quad \forall k \in S_\sigma, \end{array} \right.$$

which is the integer program $\mathrm{IP}_\sigma(y)$ in Corollary 3.3. This is because in the above integer program, obviously one should take

- $x_k = s_{k\sigma} - 1$ for all $k \notin S_\sigma,\ k \in M^+_\sigma$, and
- $x_k = 0$ for all $k \notin S_\sigma,\ k \in M^-_\sigma$,

which gives (6.14). $\square$

6.3. Proof of Theorem 5.1

First, notice that when c is replaced with αc (with $\alpha > 0$), then the optimal solutions of $\mathbb{P}$, LP (1.2) and the dual problem (4.1), are the same, whereas their respective optimal values are re-scaled by α.

Next, observe that

$$f_{\tilde{c}}(y) \leqslant \ln \gamma^*(y) \leqslant g_{\tilde{c}}(y). \tag{6.15}$$

Indeed,

$$\begin{aligned}\inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\, e^{\tilde{c}'x}) &\leqslant \inf_{z\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n_+} \Re(z^{y-Ax}\, e^{\tilde{c}'x}) \\ &= \inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n_+} e^{\lambda'(y-Ax)+\tilde{c}'x} \\ &= e^{\inf_{\lambda\in\mathbb{R}^m} \sup_{x\in\mathbb{R}^n_+} \lambda'(y-Ax)+\tilde{c}'x} \\ &= e^{g_{\tilde{c}}(y)}.\end{aligned}$$

Next, fix $z \in \mathbb{C}^m$ arbitrary. Let x^* be an optimal solution of $\mathbb{P}$. Then, as $y - Ax^* = 0$, we obtain

$$\sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\, e^{\tilde{c}'x}) \geqslant e^{\tilde{c}'x^*} = e^{f_{\tilde{c}}(y)}$$

and so, $\inf_{z\in\mathbb{C}^m} \sup_{x\in\mathbb{N}^n} \Re(z^{y-Ax}\, e^{\tilde{c}'x}) \geqslant e^{f_{\tilde{c}}(y)}$.

Now, let z_g be as in (4.8) with $\sigma = \sigma^*$ and $\alpha\lambda^*$ in lieu of λ^*. Recall that from (4.7), we have

$$z_g^{-A_j} e^{\tilde{c}_j} = 1 \quad \forall j \in \sigma^*, \quad g \in G(\sigma^*).$$

Therefore, for all $x \in \mathbb{N}^n$,

$$\begin{aligned} z_g^{y-Ax} e^{\tilde{c}'x} &= z_g^{y-\sum_{k\notin\sigma^*} A_k x_k} e^{\sum_{k\notin\sigma^*} \tilde{c}_k x_k} \prod_{j\in\sigma^*} \left(z_g^{-A_j} e^{\tilde{c}_j}\right)^{x_j} \\ &= z_g^{y-\sum_{k\notin\sigma^*} A_k x_k} e^{\sum_{k\notin\sigma^*} \tilde{c}_k x_k}\end{aligned}$$

and so,

$$\Re(z_g^{y-Ax} e^{\tilde{c}'x}) = e^{\alpha y'\lambda^*}\, e^{\sum_{k\notin\sigma^*} \alpha(c_k - A_k'\lambda^*)x_k} \times \cos 2\pi\theta_g' \left(y - \sum_{k\notin\sigma^*} A_k x_k \right).$$

From this, we can deduce that

$$\begin{aligned}\sup_{x\in\mathbb{N}^n} \Re(z_g^{y-Ax} e^{\tilde{c}'x}) = e^{\alpha y'\lambda^*} \sup_{x_k\in\mathbb{N}, k\notin\sigma^*} \Bigg[& e^{\sum_{k\notin\sigma^*} \alpha(c_k - A_k'\lambda^*)x_k} \\ & \times \cos 2\pi\theta_g' \left(y - \sum_{k\notin\sigma^*} A_k x_k \right)\Bigg].\end{aligned} \tag{6.16}$$

Let x^* be an optimal solution of the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$. We claim that for $g = g^*$ (with g^* as in Theorem 5.1), the sup in (6.16) is attained at x^* with value equal to the optimal value of $\mathbb{P}$. Suppose not. Then there exists $x_k \in \mathbb{N}$ for all $k \notin \sigma^*$, such that

$$\sum_{k\notin\sigma^*} (c_k - A_k'\lambda^*)x_k > \sum_{k\notin\sigma^*} (c_k - A_k'\lambda^*)x_k^* =: -\rho^* \tag{6.17}$$

(with $\rho^* > 0$ because $c_k - A_k'\lambda^* < 0$, $k \notin \sigma^*$), and

$$\cos 2\pi\theta_{g^*}' \left(y - \sum_{k\notin\sigma^*} A_k x_k \right) > e^{\sum_{k\notin\sigma^*} \alpha(c_k - A_k'\lambda^*)(x_k^* - x_k)}. \tag{6.18}$$

In addition, in view of our choice of g^*,

$$\cos\, 2\pi\theta'_{g^*}\left(y - \sum_{k\notin\sigma^*} A_k x_k\right) < 1$$

because if $\cos\, 2\pi\theta'_{g^*}(y - \sum_{k\notin\sigma^*} A_k x_k) = 1$ then $(y - \sum_{k\notin\sigma^*} A_k x_k) \in \oplus_{j\in\sigma^*} A_j\mathbb{Z}$, and x would be an admissible solution of the group-relaxation $\mathrm{IP}_{\sigma^*}(y)$, in contradiction with the optimality of x^*.

Now, observe that $\cos\, 2\pi\theta'_{g^*}(y - \sum_{k\notin\sigma^*} A_k x_k)$ takes finitely many values (and in fact, at most $\mu(\sigma^*)$ different values), because $(y - \sum_{k\notin\sigma^*} A_k x_k) \in \mathbb{Z}^m$. Thus,

$$1 > \delta := \max\{\cos(2\pi\theta'_{g^*} v) \mid v \in \mathbb{Z}^m;\, v \notin \oplus_{j\in\sigma^*} A_j\mathbb{Z}\}.$$

Moreover, from (6.17)

$$x_k \leqslant \sup_{k\notin\sigma^*} \rho^*/(A'_k\lambda^* - c_k) =: \beta, \quad k \notin \sigma^*. \tag{6.19}$$

Hence,

$$1 > \delta > \mathrm{e}^{\sum_{k\notin\sigma^*} \alpha(c_k - A'_k\lambda^*)(x^*_k - x_k)}. \tag{6.20}$$

So, as x_k is bounded by β, one obtains a contradiction in (6.20) when α is sufficiently small.

This proves (a) and (b). □

Appendix A.

A.1. Auxiliary result

Let $e : \mathbb{R} \to \mathbb{C}$ be the function $x \mapsto e(x) := \mathrm{e}^{2\mathrm{i}\pi x}$. First note that for all $m \in \mathbb{Z}$, $s \in \mathbb{N}$, we have the identity

$$\sum_{k=1}^{s} e(mk/s) = \begin{cases} s & \text{if } m = 0 \bmod s, \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

But we also have the following result:

Lemma A.1. *Let $m \in \mathbb{N}$ and $\{z_j\}_{j=1,\dots,m} \subset \mathbb{C}$ be the roots of $z^m - 1 = 0$. Then for all $k = 1, \dots, m-1$*

$$\begin{aligned}\sum_{\substack{1\leqslant i_1<i_2\cdots<i_k\leqslant m \\ i_1,\dots,i_k\neq j}} z_{i_1}\cdots z_{i_k} &= \sum_{\substack{1\leqslant i_1<i_2\cdots<i_k\leqslant m \\ i_1,\dots,i_k\neq j}} e((i_1+\cdots+i_k)/m) \\ &= (-1)^k z_j^k \;=\; (-1)^k\, e(kj/m).\end{aligned} \tag{A.2}$$

Proof. The proof is by induction. For $k = 1$ we have

$$\sum_{\substack{1\leqslant i\leqslant m \\ i\neq j}} z_i = \sum_{i=1}^{m} z_i - z_j \;=\; 0 - z_j,$$

because the z_j's are roots of $z^m - 1 = 0$. Next, assume that (A.2) holds for $l = 1, \dots, k$. Then as the z_j's are roots of $z^m - 1 = 0$ we have

$$\sum_{1\leqslant i_1<i_2\cdots<i_{k+1}\leqslant m} z_{i_1}\cdots z_{i_{k+1}} = 0.$$

Hence,

$$\begin{aligned}
\sum_{\substack{1\leqslant i_1<i_2\cdots<i_{k+1}\leqslant m\\ i_1,\ldots,i_{k+1}\neq j}} z_{i_1}\cdots z_{i_{k+1}} &= \sum_{1\leqslant i_1<i_2\cdots<i_{k+1}\leqslant m} z_{i_1}\cdots z_{i_{k+1}}\\
&\quad - \sum_{\substack{1\leqslant i_1\cdots i_k\leqslant m\\ i_1,\ldots,i_k\neq j}} z_j z_{i_1}\cdots z_{i_k}\\
&= 0 - z_j \times \sum_{\substack{1\leqslant i_1\cdots i_k\leqslant m\\ i_1,\ldots,i_k\neq j}} z_{i_1}\cdots z_{i_k}\\
&= -z_j(-1)^k z_j^k \quad \text{[by the induction hypothesis]}\\
&= (-1)^{k+1} z_j^{k+1}. \qquad \square
\end{aligned}$$

A.2. Some properties of the characters of $G(\sigma)$

With $\sigma \in \mathscr{B}(\Delta)$, let $G(\sigma)$ be the group $(\oplus_{j\in\sigma} A_j\mathbb{Z})^*/\Lambda^*$, of order $\mu(\sigma) =: s$, and write

$G(\sigma) = \{g_1, \ldots, g_s\}$.

Let $y \in \Lambda$ with $y \notin \oplus_{j\in\sigma} A_j\mathbb{Z}$ and consider the character $\mathrm{e}^{2\mathrm{i}\pi y}$ of $G(\sigma)$. Then $\mathrm{e}^{2\mathrm{i}\pi y}(g) = e(-g' A_\sigma^{-1} y) = e(v_g/s)$ for some $v_g \in \mathbb{N}$, $v_g < s$. That is, the mapping $g \mapsto \mathrm{e}^{2\mathrm{i}\pi y}(g)$ sends the group $G(\sigma)$ into a subgroup of the multiplicative group of the s-roots of unity. Let $s_y < s$ (with $s = p_y s_y$ for some $p_y \in \mathbb{N}$) be the order of this subgroup (which consists of the roots $\{e(j/s_y)\}$, $j = 1, \ldots, s_y$). Equivalently, s_y is the smallest integer such that $y s_y \in \oplus_{j\in\sigma} A_j\mathbb{Z}$. We can define a partition of $G(\sigma)$ (which depends on y) into s_y equivalence classes $\{C_i^y\}_{i=1}^{s_y}$ of the same cardinality $p_y := s/s_y$, by setting

$$g \sim g' \;\Leftrightarrow\; \mathrm{e}^{2\mathrm{i}\pi y}(g) = \mathrm{e}^{2\mathrm{i}\pi y}(g'), \quad g, g' \in G(\sigma). \tag{A.3}$$

We next denote by $\tilde{g}_i$ a representative of the class C_i^y and by $G_y(\sigma)$ the set $\{C_1^y, \ldots, C_{s_y}^y\}$ of equivalence classes.

We have the following result:

Lemma A.2. *Let $y \in \Lambda$ with $y \notin \oplus_{j\in\sigma} A_j\mathbb{Z}$, and let $\{C_i^y\}$ be the equivalence classes defined by* (A.3). *Then*:

(a) *For all $j \in \mathbb{N}$ with $j < s_y$, we have*

$$\sum_{1\leqslant i_1<i_2\cdots<i_j\leqslant s_y} \mathrm{e}^{2\mathrm{i}\pi y}(\tilde{g}_{i_1} + \cdots + \tilde{g}_{i_j}) = 0. \tag{A.4}$$

(b) *For all $q \in \{1, \ldots, s_y\}$ and $j < s_y$,*

$$\sum_{\substack{1\leqslant i_1<i_2\cdots<i_j\leqslant s_y\\ i_1,\ldots,i_j\neq q}} \mathrm{e}^{2\mathrm{i}\pi y}(\tilde{g}_{i_1} + \cdots + \tilde{g}_{i_j}) = (-1)^j \mathrm{e}^{2\mathrm{i}\pi y j}(\tilde{g}_q). \tag{A.5}$$

Proof. As $\mathrm{e}^{2\mathrm{i}\pi y}(\tilde{g}_i) = e(i/s_y)$ for all $i = 1, \ldots, s_y$, (a) and (b) follow from Lemma A.1. $\square$

We also have

Lemma A.3. *Let $y \in \Lambda$. Then*:

$$\sum_{g\in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(g) = \begin{cases} \mu(\sigma) & if\;\; y \in \oplus_{j\in\sigma} A_j\mathbb{Z},\\ 0 & \text{otherwise}\end{cases} \tag{A.6}$$

and

$$\prod_{\tilde{g}\in G_y(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(\tilde{g}) = (-1)^{s_y+1}. \tag{A.7}$$

Proof. If $y \in \oplus_{j\in\sigma} A_j\mathbb{Z}$ then $\mathrm{e}^{2\mathrm{i}\pi y}(g) = 1$ for all $g \in G(\sigma)$, which yields the first part of (A.6). On the other hand, if $y \notin \oplus_{j\in\sigma} A_j\mathbb{Z}$ we proceed as before. Let $G_y(\sigma)$ be the set of s_y equivalence classes of $G(\sigma)$ defined in (A.3). We thus have

$$\sum_{g\in G(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(g) = p_y \sum_{\tilde{g}\in G_y(\sigma)} e^{2\mathrm{i}\pi y}(\tilde{g}) = p_y \sum_{j=1}^{s_y} e(j/s_y) = 0,$$

which proves (A.6). Next,

$$\prod_{\tilde{g}\in G_y(\sigma)} \mathrm{e}^{2\mathrm{i}\pi y}(\tilde{g}) = \prod_{j=1}^{s_y} e(j/s_y)$$
$$= (-1)^{s_y+1},$$

which proves (A.7). $\square$

References

[1] K. Aardal, R. Weismantel, L.A. Wolsey, Non-standard approaches to integer programming, Discrete Appl. Math. 123 (2002) 5–74.
[2] T.V. Alekseevskaya, I.M. Gel'fand, A.V. Zelevinsky, An arrangement of real hyperplanes and the partition function connected with it, Soviet Math. Dokl. 36 (1988) 589–593.
[3] A. Barvinok, J.E. Pommersheim, An algorithmic theory of lattice points in polyhedra, in: L.J. Billera et al., (Ed.), New Perspectives in Algebraic Combinatorics Vol. 38, MSRI Publications, 1999, pp. 91–147.
[4] C.E. Blair, R.G. Jeroslow, The value function of an integer program, Math. Programming 23 (1982) 237–273.
[5] M. Brion, M. Vergne, Residue formulae, vector partition functions and lattice points in rational polytopes, J. Amer. Math. Soc. 10 (1997) 797–833.
[6] B. Chen, Lattice points, Dedekind sums, and Ehrhart polynomials of lattice polyhedra, Discrete Comput. Geom. 28 (2002) 175–199.
[7] P. Conti, C. Traverso, Buchberger algorithm and integer programming, in: F.F. Mattson, T. Mora, T.R.N. Rao (Eds.), Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, Lecture Notes in Computer Science, Vol. 539, Springer, Berlin, 1991, pp. 130–139.
[8] R.E. Gomory, Some polyhedra related to combinatorial problems, Linear Algebra Appl. 2 (1969) 451–558.
[9] S. Hosten, R. Thomas, Groebner bases and integer programming, in: B. Buchberger, F. Winkler (Eds.), Groebner Bases and Applications, Proceedings of 33 Years of Groebner Bases Conference, London Mathematical Society Lecture Note Series 251, Cambridge University Press, Cambridge, 1998, pp. 144–158.
[10] S. Hosten, R. Thomas, Gomory integer programs, Math. Programming 96B (2003) 271–292.
[11] J.B. Lasserre, La valeur optimale des programmes entiers, C.R. Acad. Sci. Paris Ser. I 353 (2002) 863–866.
[12] J.B. Lasserre, Duality and a Farkas lemma for integer programs, in: E. Hunt, C.E.M. Pearce (Eds.), Optimization: Structure and Applications, Applied Optimization Series, Kluwer Academic Publishers, 2004.
[13] G.L. Nemhauser, Duality for integer optimization, in: M. O 'hEigeartaigh, J.K. Lenstra, A.H.G. Rinnooy Kan (Eds.), Combinatorial Optimization: Annotated Bibliographies, Wiley, Chichester, 1985.
[14] G.L. Nemhauser, L.A. Wolsey, Integer and Combinatorial Optimization, Wiley, New York, 1988.
[15] A.V. Pukhlikov, A.G. Khovanskii, A Riemann–Roch theorem for integrals and sums of quasipolynomials over virtual polytopes, St. Petersburg Math. J. 4 (1993) 789–812.
[16] A. Schrijver, Theory of Linear and Integer Programming, Wiley, Chichester, 1986.
[17] B. Sturmfels, R.R. Thomas, Variation of cost functions in integer programming, Math. Programming 77 (1997) 357–387.
[18] R.R. Thomas, A geometric Buchberger algorithm for integer programming, Math. Oper. Res. 20 (1995) 864–884.
[19] R.R. Thomas, Algebraic methods in integer programming, in: C. Floudas, P. Pardalos (Eds.), Encyclopedia of Optimization, Kluwer Academic Publishers, Dordrecht, 2001.
[20] L. Wolsey, Extensions of the group theoretic approach in integer programming, Management Sci. 18 (1971) 74–83.
[21] L.A. Wolsey, Integer programming duality: price functions and sensitivity analysis, Math. Programming 20 (1981) 173–195.